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Application of the "Generalized Riemann Problem" Method to 1-D Compressible Flows with Material Interfaces*

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The "Generalized Riemann Problem" (GRP) method is applied to 1-D compressible flows with material interfaces and variable cross section. The resulting scheme is second-order and uses a "mixed-type" grid, where cell boundaries can be either Lagrangian or Eulerian. In fact, using the analytic resolution of discontinuities at cell boundaries, provided by the GRP solution, one can extend the scheme presented here to include any adaptive mesh.

Two numerical examples are studied: a planar shock-tube and exploding helium sphere. It is shown that discontinuities are sharply resolved while there are no oscillations in the smooth part of the flow. In particular, wave interactions, including formation of new shocks and reflection from the center of symmetry, are automatically taken care of. © 1986 Academic Press, Inc.

1. INTRODUCTION

In two recent papers [1, 2] a second-order upwind scheme was presented for the computation of time-dependent inviscid compressible fluid flow in one space dimension and variable cross section. This scheme has been successfully applied to a variety of test-cases, involving single-material (Eulerian or Lagrangian) flows, yielding always sharp resolution of discontinuities and very smooth solutions between them.

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However, in physical applications one often encounters multi-material flows. To avoid diffusive effects in Eulerian calculations, it is necessary to keep track of the fluid interfaces. The introduction of interfaces gives rise to computational cells of "mixed type," namely, cells which have one Eulerian and one Lagrangian endpoint.

In the present work we show that the original GRP (Generalized Riemann Problem) scheme [1, 2] can be readily extended to allow for such a situation, using analytic expressions for the fluxes between cells.

It is important to note that the scheme presented here can be used with any *adaptive mesh*. In fact, the fluxes can be analytically evaluated across any line emanating from the singularity (representing the jump between two adjacent cells). This enables one to follow shock trajectories or any other "preferred" directions in the grid.

The paper consists of two parts. In the next section we discuss the scheme in general terms. The various quantities are reduced to those given explicitly in [2], so we feel it is not needed to give again the explicit formulae. In Section 3 we discuss two numerical examples. The first is the well-known shock-tube test problem suggested by Sod [4]. We compare the results of the pure Eulerian GRP method and the present one and demonstrate how the resolution of the contact discontinuity is improved while the other regions of the flow remain intact. The second example deals with the case of a helium sphere exploding into air. We have taken the data from a paper by Glass and Saito [3]. This case involves a complex wave structure, including the formation of a new (converging) shock, reflections from the center of symmetry, etc.

It should be emphasized that both test cases were run on the same code, using a "minimal" monotonicity algorithm. Also, no special treatment was needed at the center of symmetry when we implemented there the procedure discussed in [2].

2. Discussion of the Scheme

Consider the system of equations representing time-dependent inviscid compressible fluid flow in one space dimension r but with variable cross section $A(r)$. In "quasi-conservation" form, these equations are

$$A\frac{\partial}{\partial t}U+\frac{\partial}{\partial r}[AF(U)]+A\frac{\partial}{\partial r}G(U)=0 \tag{2.1}$$

$$U=\begin{pmatrix}\rho\\ \rho u\\ \rho E\end{pmatrix},\qquad F(U)=\begin{pmatrix}\rho u\\ \rho u^2\\ (\rho E+p)\,u\end{pmatrix},\qquad G(U)=\begin{pmatrix}0\\ p\\ 0\end{pmatrix}.$$

Here ρ, p, u are, respectively, density, pressure, and velocity, $E=e+\frac{1}{2}u^2$ is the total specific energy (e being the internal specific energy), and an equation of state of the form $p=p(e,\rho)$ is assumed.

Now let D_t be a moving zone in the flow given by

$$D_t = \{r \mid a(t) \leqslant r \leqslant b(t)\}.$$

Clearly,

$$\begin{aligned}\frac{d}{dt}\int_{D_t} UA(r)\,dr = \int_{D_t} \frac{\partial U}{\partial t} A(r)\,dr + b'(t)\,A(b(t))\,U(b(t), t)&\\ - a'(t)\,A(a(t))\,U(a(t), t).&\end{aligned} \tag{2.2}$$

In conjunction with Eq. (2.1) we get, using obvious notation,

$$\frac{d}{dt}\int_{D_t} UA(r)\,dr = [A(r)(UA - F(U))]_{a(t),t}^{b(t),t} - \int_{D_t} \frac{\partial}{\partial r} G(U)\cdot A(r)\,dr, \tag{2.3}$$

where $A(r(t), t) = r'(t)$.

As usual, a finite difference form of (2.3) is obtained by selecting a sequence $t_n = n\,\Delta t$ of t-values and replacing the continuous values of U by discrete cell averages U_i^n. Here the index i denotes the spatial cell with the moving boundaries a_i^n, b_i^n ($a_i^n \leqslant r \leqslant b_i^n$), whose volume is given by

$$V_i^n = \int_{a_i^n}^{b_i^n} A(r)\,dr.$$

Values at the cell-boundaries a_i, b_i are labeled, respectively, by $i - \frac{1}{2}$ and $i + \frac{1}{2}$. In order to obtain a time-centered scheme they must be evaluated at $t_{n+1/2}$. Thus, our finite difference form corresponding to (2.3) is

$$\begin{aligned}U_i^{n+1} = U_i^n \frac{V_i^n}{V_i^{n+1}} + \frac{\Delta t}{V_i^{n+1}} \{&[(UA - F(U))\,A]_{i+1/2}^{n+1/2} - [(UA - F(U))\,A]_{i-1/2}^{n+1/2}\\ &- \frac{1}{2}(G(U)_{i+1/2}^{n+1/2} - G(U)_{i-1/2}^{n+1/2})(A_{i+1/2}^{n+1/2} + A_{i-1/2}^{n+1/2}\}.\end{aligned} \tag{2.4}$$

In order to obtain second-order accuracy for the scheme (2.4) we assume that the values U^n are *linearly* distributed in cells. The boundary values $U_{i+1/2}^{n+1/2}$ are then determined by the GRP analytic method of [2]. Note that cell-boundaries are moving relative to the Eulerian grid, so that the full strength of the GRP method is needed in order to determine the time behavior of flow quantities along curves emanating from the singularity. In our examples the "mixed-type" cells had one Lagrangian (namely, $A = u$) and one Eulerian ($A = 0$) boundary. In both cases, a detailed analysis of the flux-vectors was given in [2] and we shall not repeat it here. Finally, the slopes at time t_{n+1} are determined simply by $(U_{i+1/2}^{n+1} - U_{i-1/2}^{n+1})/(r_{i+1/2}^{n+1} - r_{i-1/2}^{n+1})$. These slopes are subject to van Leer's monotonicity algorithm, as discussed in [1].

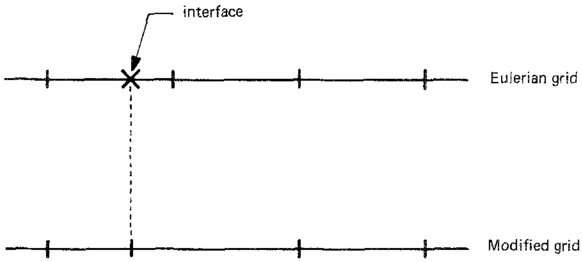

FIG. 1. The Eulerian grid modified by an interface point ($i \pm \frac{1}{2}$ denote boundaries of cell i).

In implementing the scheme (2.4) one must take into account one additional feature which is associated with the moving boundaries, namely, the varying sizes of computational cells. In particular, a moving cell-boundary may approach its neighboring Eulerian grid-point, thus shrinking one computational cell and expanding the other. This would force a very low time step and an inefficient and distorted calculation. To avoid such phenomena, and at the same time keep the programming simple and robust, we have used the following procedure. Each interface point (that is, a point designated to be a Lagrangian moving boundary) replaces its nearest Eulerian grid-point. Thus, a pair of adjacent "pure" Eulerian cells (one of which contains the interface) is replaced by a pair of "mixed-type" cells having the interface point as a common boundary and a total length of $2\,\Delta r$ (see Fig. 1).

Whenever the (Eulerian) length of a "mixed-type" cell exceeds $1.5\,\Delta r$, the eliminated Eulerian grid-point is restored, and the one on the other side of the interface is eliminated. Thus, lengths of "mixed-type" cells are always between $0.5\,\Delta r$ and $1.5\,\Delta r$. This procedure of elimination and restoration of Eulerian grid-points conserves mass, momentum, and total energy and enables moving boundaries to travel across the Eulerian grid without distorting considerably the mesh.

3. Numerical Examples

(a) Our first example is the planar shock-tube test problem suggested by Sod [4]. The tube extends from $x = 0$ to $x = 100$ and is divided into 100 equal cells. The gas is initially at rest ($u = 0$) with $p = \rho = 1$ for $0 \leqslant x < 50$, $p = 0.1$, $\rho = 0.125$ for $50 < x \leqslant 100$. The resulting (self-similar) solution involves a shock moving to the right and a centered rarefaction wave travelling to the left. They are separated by a contact discontinuity. The pure Eulerian version of the GRP method has been applied to this problem in [1]. In Fig. 2 we show the results of this computation at $t = 15$, when the shock has moved approximately 25 points to the right. Note that there is some smearing (2–3 mesh-points) of the contact discontinuity. In Fig. 3 we show profiles (velocity, pressure, density, and p/ρ^{γ}, $\gamma = 1.4$) for the same problem, using the present "mixed" scheme. As could be expected, the contact discontinuity

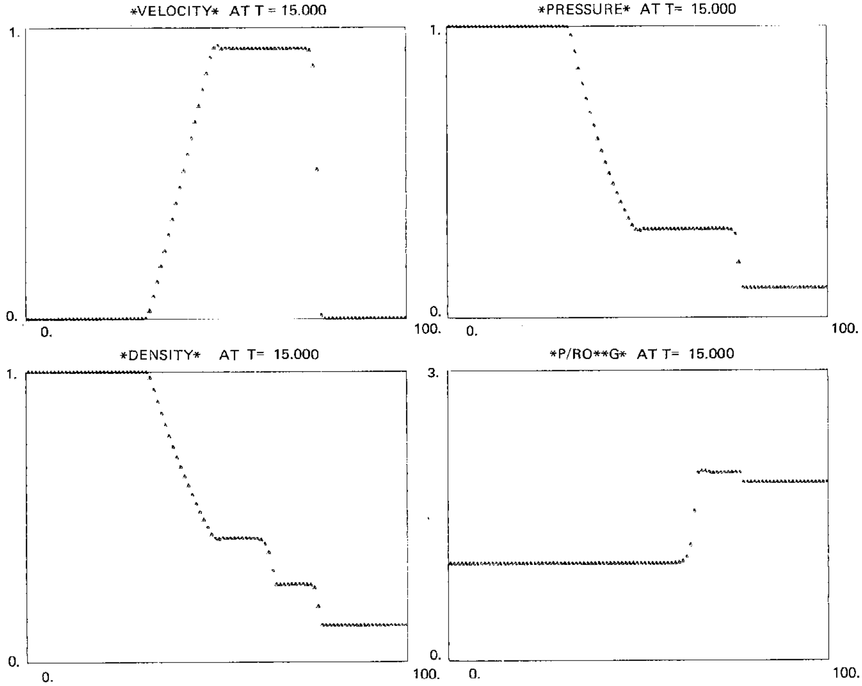

FIG. 2. Results for Sod's problem, using pure Eulerian grid.

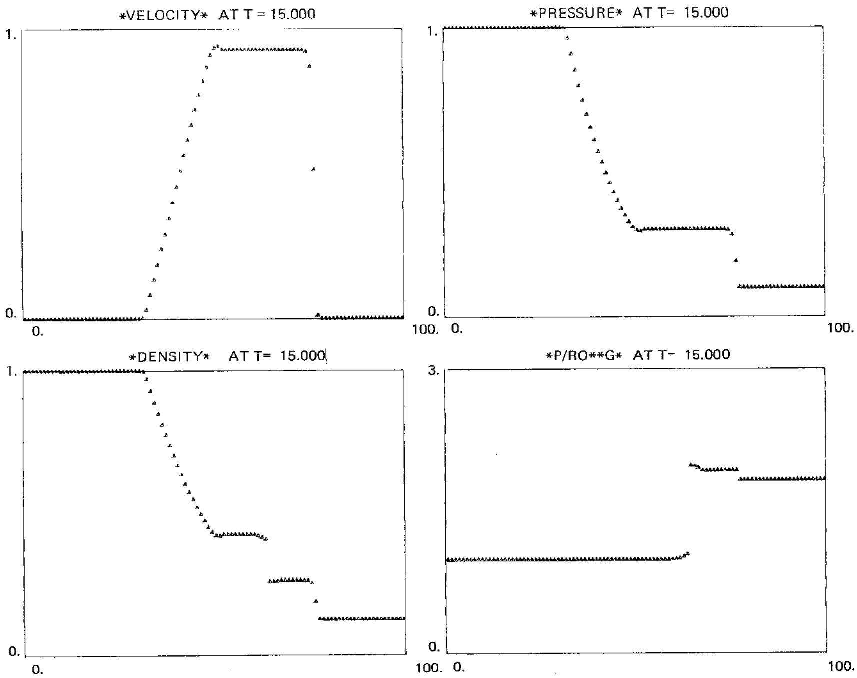

FIG. 3. Results for Sod's problem, using the modified grid.

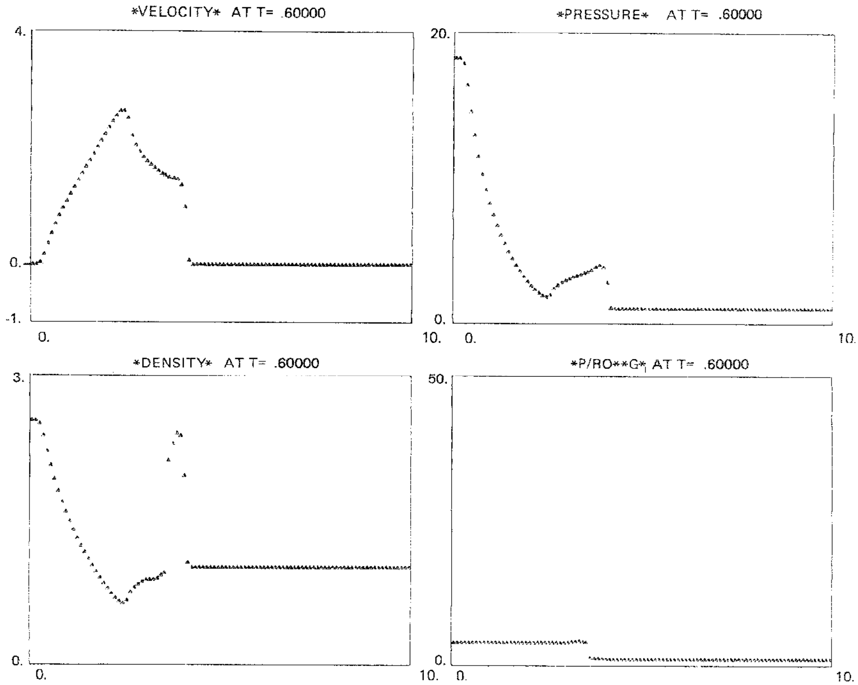

FIG. 4. Explosion of helium sphere, profiles at $t = 0.6$.

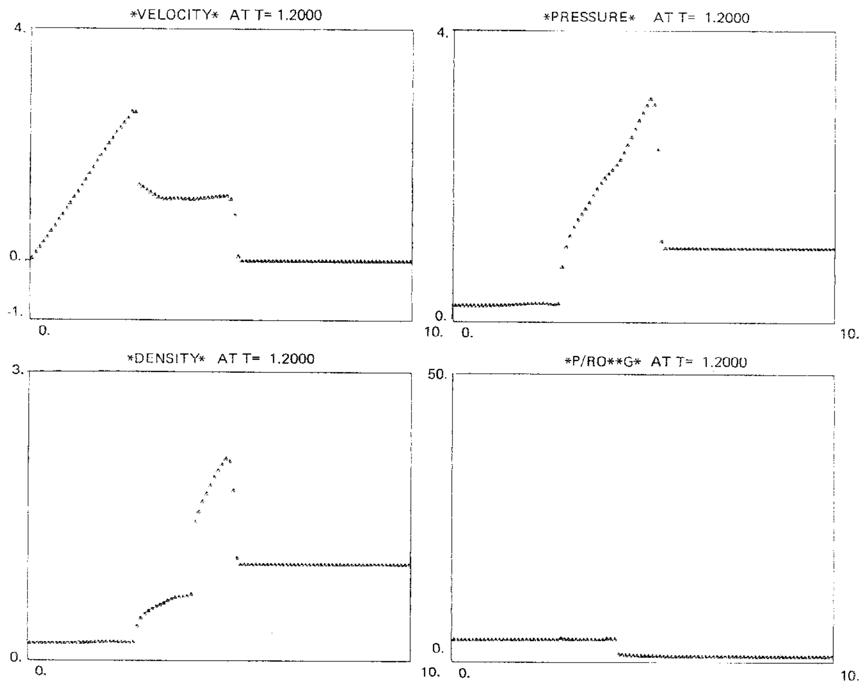

FIG. 5. Explosion of helium sphere, profiles at $t = 1.2$.

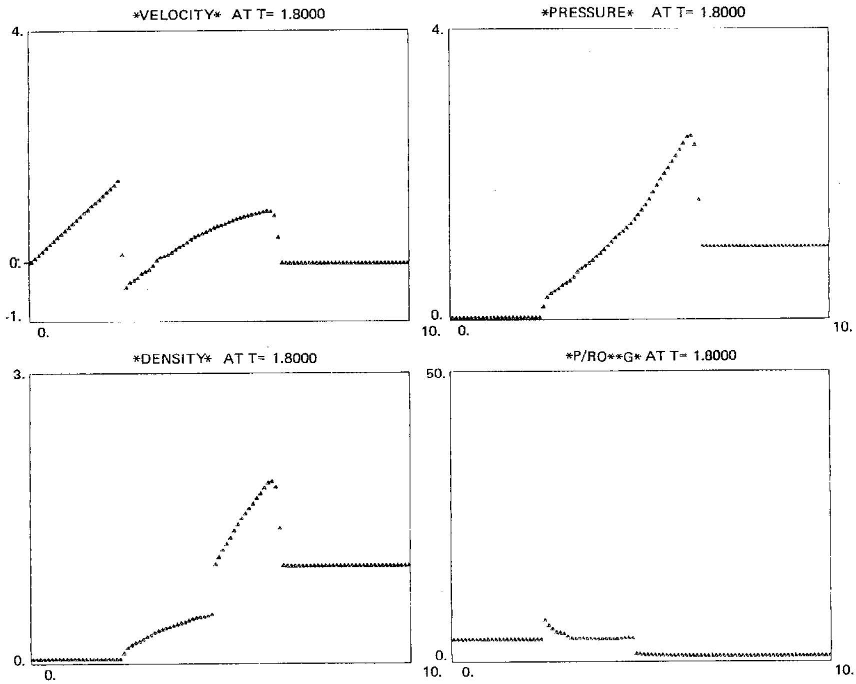

FIG. 6. Explosion of helium sphere, profiles at $t = 1.8$.

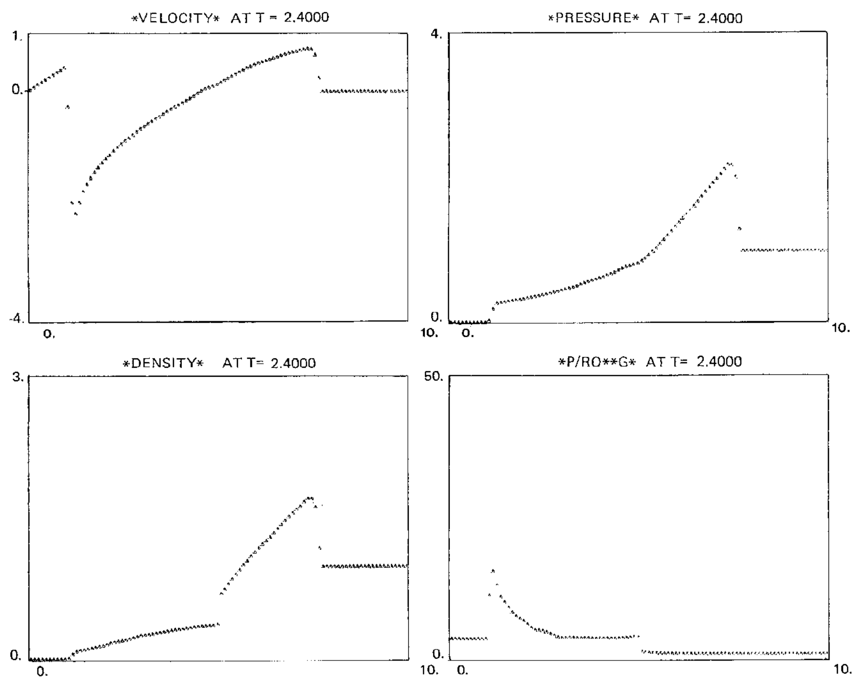

FIG. 7. Explosion of helium sphere, profiles at $t = 2.4$.

is sharply resolved. At the same time, the scheme maintains the high-resolution features of other regions of the flow (shock front, edges of rarefaction).

(b) To demonstrate our scheme in a case involving two materials and a variable cross section, we took up a test case involving a pressurized helium sphere exploding into air. The resulting spherically symmetric flow has been computed previously by Glass and Saito using a modified random choice method [3], which seems to have been the only method capable of resolving the complex wave structure in this case. A helium ($\gamma = 5/3$) sphere of radius 2.5 is surrounded by air ($\gamma = 7/5$) and is initially at rest with uniform pressure which is 18.25 times the surrounding air pressure and uniform density which is 2.523 times that of the air. In Figs. 4–8 we plot the profiles of flow variables at 0.6 time unit intervals (for a sphere of 2.5 cm, air pressure of 1 atmosphere, and air density of $1.29 \cdot 10^{-3}$ gr/cm^3, one time unit is 35.68 μsec). Let us review briefly the flow evolution in this case (see [3] for a more detailed discussion).

At $t = 0.6$ (Fig. 4) we observe the diverging shock, followed by a contact discontinuity and a very strong rarefaction. In fact, the excessive rarefaction leads to the formation of an inward-facing shock which is clearly seen at $t = 1.2$ (Fig. 5).

By that time, the rarefaction has been fully reflected from the center of symmetry

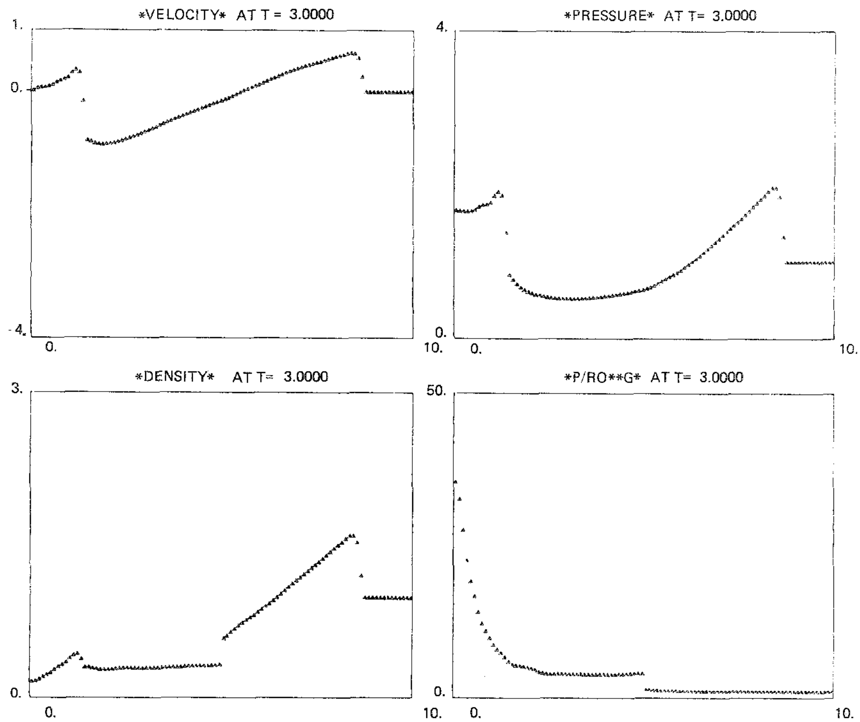

FIG. 8. Explosion of helium sphere, profiles at $t = 3.0$.

and the velocity profile is nearly linear, as could be expected for such a solution with self-similarity.

At times 1.8 and 2.4 (Figs. 6 and 7) it is clearly seen that, while the diverging shock decelerates, the converging shock gains speed, leading to a large velocity gradient behind it. We have located the time of arrival of this shock at the center around $t = 2.6$. This is in agreement with the results of [3].

Finally, at $t = 3$ (Fig. 8) both shocks are diverging. Notice the sharp resolution of the contact discontinuity and the overall smoothness of the flow profiles. In fact, studying the time evolution of the flow near $r = 0$ following the reflection of the rarefaction wave (i.e., $t > 0.6$), we could fit our numerical results by

$$u(r, t) \sim r/t, \qquad p(0, t) \sim t^{-5.5}, \qquad \rho(0, t) \sim t^{-3.3}.$$

This result is in complete agreement with the isentropic nature of the flow there, namely, p/ρ^{γ} is constant, with $\gamma = 5/3$.

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